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Hamiltonian and gradient structures in the Toda flows

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Abstract

In this paper we consider gradient structures in the dynamics and geometry of the asymmetric nonperiodic tridiagonal and full Toda flow equations. We compare and contrast a number of formulations of the nonperiodic Toda equations. In the case of the full Kostant (asymmetric) Toda flow we explain the role of noncommutative integrability in its qualitative behavior. We describe the relationship between the asymmetric Toda flows and the symmetric and indefinite Toda flows, and prove in particular that one may conjugate from the full Kostant Toda flows to the full symmetric Toda flows via a Poisson map. © 1998 Elsevier Science B.V.

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1. Introduction

The dynamics and geometry of the classical nonperiodic Toda lattice equations have generated a tremendous amount of research over the last couple of decades. This includes the original work of Toda [35] on the Toda lattice, the work of Flaschka [19] (see also [23]) who showed how to write the lattice flow in tridiagonal Lax pair form, and the work of Moser [29] who analyzed the finite nonperiodic Toda lattice and its spectral properties. Kostant [27] (see also [7]) generalized these results to Toda lattices associated with arbitrary semisimple Lie algebras.

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Moser showed that the nonperiodic Toda lattice equations are gradient on a level set of its integrals. Bloch [3] and Bloch et al. [5] showed that this gradient flow behavior is exhibited in the original Flaschka variables as a gradient flow with respect to the normal metric on an adjoint orbit of a compact Lie group. This flow takes the so-called double bracket form. It was also shown (see [3,6]) that the original gradient flow of Moser can be written in double bracket form as a gradient flow on a projective space. A key part of our paper here involves showing how this gradient behavior extends to more general forms of the Toda flows (see also the work of de Mari and Pedroni [16] discussed below).

An important aspect of Kostant's work was that he wrote the Toda lattice in asymmetric tridiagonal form. This turns out to be more general than the symmetric form used in much of the literature. In the current paper we show how the Kostant Toda lattice equations may be viewed as a gradient flow, and we discuss the relationship of this flow to double bracket flows on noncompact manifolds. The latter flows are a special case of the Kostant Toda flow and are related to the work of Faybusovich [18] and Kodama and Ye [24,25] on Toda flows with indefinite metric (see also [10]). We shall call flows of this type "signed" Toda flows since they are defined by prescribing a signature matrix.

We also consider here the full Toda lattice, i.e. the Toda flows on generic orbits of the coadjoint action of the Borel subgroup. The key original paper in this area is that of Deift et al. [13] (see also [34]), and this was followed by the work of Ercolani et al. [17] among others. While it is simple to conjugate from the tridiagonal asymmetric to the symmetric Toda lattice, this is much trickier in the full case, but we indicate a method for doing this. A recent paper on the full symmetric Toda lattice is that of de Mari and Pedroni [16], which discusses the gradient nature of these flows. Here we consider their analysis and its relationship with the Hamiltonian structure and level sets of the flow. We also consider the gradient nature of the nonsymmetric Kostant full Toda flow. Again, while this is apparent from its asymptotic behavior, understanding its geometry is more subtle than the symmetric case. We show that the gradient-like behavior is in fact essentially due to the noncommutative integrability (see [30]) of the full Toda flow. We give a sketch of the proof of noncommutative integrability here, but refer the reader to a related forthcoming paper [22], for further details. We also discuss briefly the "full QR" flows described in [14], where the flow is on the space of arbitrary $n \times n$ matrices – commutative integrability also holds in this setting.

The contents of the paper are as follows. In Section 2 we review the necessary background and exhibit the gradient nature of the generalized Kostant and signed Toda lattices and full Kostant Toda flows. In Section 3 we discuss the full Toda flows, showing how to conjugate the Kostant asymmetric flow to signed flows, proving this map is Poisson, and establishing the link with noncommutative integrability. In Section 4 we make some concluding remarks.

2. The gradient nature of generalized Toda flows

We begin by describing a very general (Hamiltonian) formulation of the Toda flows. We shall then consider various special cases and discuss their gradient nature.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$. Let Φ denote the system of roots of $\mathfrak{g}$ defined by $\mathfrak{h}$ and let Δ denote the simple roots. Choose $\{h_j, e_\alpha \mid j = 1, \dots, l, \alpha \in \Phi\}$ to be a Chevalley basis of $\mathfrak{g}$ with $\alpha_1, \dots, \alpha_n$ denoting the simple roots. Here h_j lie in the real part $\mathfrak{a}$ of $\mathfrak{h}$.

Let $\mathfrak{g}_n$ be the normal real form of $\mathfrak{g}$ and denote by $\mathfrak{b}_+$ its upper Borel subalgebra. Different types of Toda flows to be considered below are associated with different realizations of the dual space $\mathfrak{b}^*$ as an affine subspace of $\mathfrak{g}_n$ via the decomposition of $\mathfrak{g}_n$ into the direct sum of subalgebras

$$\mathfrak{g}_n = \mathfrak{g}_0 + \mathfrak{b}_+. \tag{2.1}$$

Let $\mathfrak{g}_0^\perp$ be the annihilator of $\mathfrak{g}_0$ with respect to the Killing form $\langle\ ,\ \rangle$. According to the Adler–Kostant–Symes formalism (see e.g. [32,33]), one identifies $\mathfrak{b}_+^*$ with $\epsilon + \mathfrak{g}_0^\perp$, where ϵ is a fixed element of $\mathfrak{b}_+^\perp \cap [\mathfrak{g}_0, \mathfrak{g}_0]^\perp$.

The generalized Toda flow on $\epsilon + \mathfrak{g}_0^\perp$ is generated by the Hamiltonian $H(L) = \frac{1}{2}\langle L, L\rangle$ in the Poisson structure obtained as a pull-back of the Lie–Poisson bracket on $\mathfrak{b}_+^*$:

$$\{f_1, f_2\}_{\epsilon+\mathfrak{g}_0^\perp}(L) = \langle L, [\pi_+ \nabla f_1(L), \pi_+ \nabla f_2(L)]\rangle, \tag{2.2}$$

where gradients are defined with respect to the Killing form and π_+ is a projection on $\mathfrak{b}_+$ parallel to $\mathfrak{g}_0$. The corresponding equations of motion have the Lax form

$$\dot{L} = [L, \pi_+(L)] = [\pi_{\mathfrak{g}_0}(L), L]. \tag{2.3}$$

Here $\pi_{\mathfrak{g}_0} = \mathrm{Id} - \pi_+$. More generally, the higher Toda flow generated by the Hamiltonian $H(L)$, where H is an invariant function on $\mathfrak{g}$, has the form

$$\dot{L} = [L, \pi_+(\nabla H(L))]. \tag{2.4}$$

As is well-known (see e.g. [20,33]), Eq. (2.4) can be integrated by means of the factorization method. Furthermore, let f be a restriction to $\epsilon + \mathfrak{g}_0^\perp$ of an Ad_{B_+}-invariant function on $\mathfrak{g}$, where B_+ is the upper Borel subgroup. Then f is an integral of motion of the Toda flow and the Poisson bracket of any two such integrals f_1, f_2 is given by

$$\{f_1, f_2\}_{\epsilon+\mathfrak{g}_0^\perp}(L) = \langle L, [\nabla f_1(L), \nabla f_2(L)]\rangle. \tag{2.5}$$

In particular, if J is a Chevalley invariant of $\mathfrak{g}$, then $\{J, f\}_{\epsilon+\mathfrak{b}_-} = 0$ for any Ad_{B_+}-invariant function f.

The above discussion provides a uniform description of Hamiltonian properties of the Toda flows associated with any decomposition (2.1). In contrast, when Bloch et al. [4,5] showed that the generalized tridiagonal Toda lattice associated with a compact semisimple Lie group describes a gradient flow on level sets of integrals, the result seemed to be a special feature of the symmetric tridiagonal case. In their recent paper, de Mari and Pedroni [16] used a modified Killing form to construct a positive definite metric, with respect to which the full generalized symmetric Toda flow is gradient. What we are about to show is that the gradient structure in Toda flows is even more general and can be observed in the nonsymmetric case too.

We recall firstly the corresponding result for the symmetric Toda flow. As we shall see, dealing with the asymmetric case requires a rather different approach.

Recall the definition of the normal metric on an orbit of a Lie group (see [2,6]):

Let $\kappa(\ ,\ ) = -\langle\ ,\ \rangle$ be the Killing form and decompose $\mathfrak{g}$ orthogonally relative to $\langle\ ,\ \rangle$ into $\mathfrak{g} = \mathfrak{g}^L \oplus \mathfrak{g}_L$, where $\mathfrak{g}_L$ is the centralizer of L and $\mathfrak{g}^L = \operatorname{Im} \operatorname{ad} L$. For $X \in \mathfrak{g}$ denote by X^L the projection of X onto $\mathfrak{g}^L$. Then, given two tangent vectors to the orbit $[L, X]$ and $[L, Y]$, the normal metric is defined by $\langle [L, X], [L, Y] \rangle_N = \langle X^L, Y^L \rangle$. Note that for an arbitrary semisimple Lie algebra this metric will be indefinite. For an orbit in a compact Lie algebra it will be definite.

It was shown in [4,5] that the following holds: Let $\mathfrak{g}_u$ denote the compact form of a complex semisimple Lie algebra.

Then we have the following result.

Proposition 2.1. *The gradient vector field of the function $H(L) = \kappa(L, N)$ on the adjoint orbit $\mathcal{O}$ of $\mathfrak{g}_u$ containing the initial condition $L(0) = L_0$ with respect to the normal metric $\langle , \rangle_N$ on $\mathcal{O}$ is given by*

$$\dot{L}(t) = [L(t), [L(t), N]]. \tag{2.6}$$

Now let $\mathfrak{A}$ be a maximal abelian subalgebra of $\mathfrak{g}_u$ and choose $\mathfrak{H} = \mathfrak{A} \oplus \mathrm{i}\mathfrak{A}$ as the Cartan subalgebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}_u$. Choose a Chevalley basis as described above, with α_j denoting the simple roots as before.

Bloch et al. [4,5] proved the following result for the symmetric tridiagonal generalized Toda flow (see [3] for the sl(n) case):

Theorem 2.2. *If N is i times the simple coweights of $\mathfrak{g}$, then for*

$$L = \sum_j^l i b_j h_j + \sum_j^l i a_j (e_{\alpha_j} + e_{-\alpha_j}), \tag{2.7}$$

Eq. (2.6) *gives the generalized* (*tridiagonal, symmetric*) *Toda lattice equations on the level set of all integrals of the Toda flow. Explicitly, N is given by*

$$N = \sum_j i x_j h_j, \tag{2.8}$$

where $(x_1, \ldots, x_l)$ is the unique solution of the system

$$\sum_j x_j \alpha_i(h_j) = -1, \quad i = 1, \ldots, l. \tag{2.9}$$

In the case of the symmetric full Toda flow, Proposition 2.1 and Theorem 2.2 generalize, as shown by de Mari and Pedroni [16] (see also [9,38] for metrics of this type) to the following:

Proposition 2.3. *Let L be given by*

$$L = \sum_{j}^{l} i b_j h_j + \sum_{\lambda \in \Phi} i a_j (X_\lambda + X_{-\lambda}), \tag{2.10}$$

where X_λ is a root vector with weight λ and let N be defined as before. Let J be the symmetric positive definite operator that assigns to an element of $\mathfrak{g}_u$ zero if it lies in the Cartan subalgebra corresponding to the given Chevalley basis and multiplies it by the inverse of its weight otherwise. Then the gradient vector field of the function $H(L) = \kappa(L, N)$ on the adjoint orbit $\mathcal{O}$ of $\mathfrak{g}_u$ containing the initial condition $L(0) = L_0$ with respect to the modified normal metric $\langle \cdot, J\cdot \rangle_N$, on $\mathcal{O}$ is given by

$$\dot{L}(t) = [L(t), J[L(t), N]] \tag{2.11}$$

and gives the full symmetric Toda flow.

The idea of the proof is simply that J "cancels out" the weight assigned by commuting with N.

2.1. The signed Toda lattice

Faybusovich [18] and Kodama and Ye [24] considered *the signed Toda flows* in the form

$$\dot{L}J = [LJ, \pi_s(LJ)], \tag{2.12}$$

where L is a tridiagonal or full symmetric matrix and

$$J = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_i = \pm 1, \tag{2.13}$$

is a signature matrix.

Denote

$$\mathrm{su}(J) = \{X \in \mathrm{sl}(n, \mathbb{C}) \colon JX = -X^* J\}, \tag{2.14}$$

$$\mathrm{so}(J) = \{X \in \mathrm{sl}(n, \mathbb{R}) \colon JX = -X^t J\}, \tag{2.15}$$

$$S(J) = \{X \in \mathrm{sl}(n, \mathbb{R}) \colon JX = X^* J\}. \tag{2.16}$$

It can be easily checked that for L symmetric, $LJ \in S(J)$ and $\mathrm{i}(LJ - \mathrm{Tr}(LJ)Id) \in \mathrm{su}(J)$.

This suggests the following extension of the definition of the signed Toda flow to an arbitrary semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_\tau$ be a real form of $\mathfrak{g}$ that corresponds to the positive eigenspace of an antilinear involutive automorphism τ. Assume that $\mathfrak{g}_\tau$ is compatible with the normal real form $\mathfrak{g}_n$. Then (see, e.g. [31, Chapter 5])

$$\mathfrak{g}_n = (\mathfrak{g}_n) \cap (\mathfrak{g}_\tau) + (\mathfrak{g}_n) \cap (\mathrm{i}\mathfrak{g}_\tau).$$

We assume additionally, that there exists a direct sum decomposition

$$\mathfrak{g}_n = (\mathfrak{g}_n) \cap (\mathfrak{g}_\tau) + \mathfrak{b}, \tag{2.17}$$

where $\mathfrak{b}$ is the upper Borel subalgebra in $\mathfrak{g}_n$.

Then, we have a decomposition (2.1) with $\mathfrak{g}_0 = (\mathfrak{g}_n) \cap (\mathfrak{g}_\tau)$. One can choose $\epsilon = 0$ and consider the corresponding Toda flow (2.3) on $\mathfrak{g}_0^\perp = (\mathfrak{g}_n) \cap (\mathrm{i}\mathfrak{g}_\tau)$.

In particular, if we choose τ to be the unique antilinear involutive automorphism such that

$$\tau(e_{\alpha_i}) = \mu_i e_{-\alpha_i}, \qquad \tau(e_{-\alpha_i}) = \mu_i e_{\alpha_i}, \qquad \tau(h_i) = -h_i, \tag{2.18}$$

where $\mu_i = \pm 1$, then the flow (2.3) can be restricted to the subspace of "tridiagonal" elements in $(\mathfrak{g}_n) \cap (\mathrm{i}\mathfrak{g}_\tau)$ having the form

$$L = \sum_j^l b_j h_j + \sum_j^l a_j (e_{\alpha_j} - \mu_j e_{-\alpha_j}). \tag{2.19}$$

The projection $\pi_\tau(L)$ is then determined to be

$$\pi_{\mathfrak{g}_0}(L) = -\sum_j^l a_j (e_{\alpha_j} + \mu_j e_{-\alpha_j}). \tag{2.20}$$

We call the corresponding Lax equation (2.3) *the generalized signed Toda lattice*. Note that if $\mathfrak{g}$ is $\mathrm{sl}(n)$ and τ is defined by (2.18) with $\mu_i = -\epsilon_i \epsilon_{i+1}$, then Eq. (2.3) takes the form (2.12).

A direct computation shows that

$$\pi_{\mathfrak{g}_0}(L) = -\mathrm{i}\,[L, N],$$

where N is defined by (2.8) and (2.9). Furthermore, it follows from (2.18) and (2.19) that $\mathrm{i}L$ belongs to $\mathfrak{g}_\tau$. Therefore, the generalized signed Toda lattice can be written in the double bracket form (2.6). Moreover, one can modify the argument from Bloch et al. [4,5] in order to prove the following analog of Proposition 2.1 and Theorem 2.2.

Theorem 2.4. *The generalized signed Toda lattice describes the gradient flow of the function $H(\mathrm{i}L) = \kappa(\mathrm{i}L, N)$ on the adjoint orbit $\mathcal{O}$ of $\mathfrak{g}_\tau$ containing the initial condition $\mathrm{i}L(0) = \mathrm{i}L_0$ with respect to the (indefinite) normal metric $\langle\ ,\ \rangle_N$ on $\mathcal{O}$.*

2.2. *The generalized Kostant Toda lattice*

Following Kostant [27], we now choose in (2.1) $\mathfrak{g}_0$ to be equal to $\mathfrak{n}$, the nilradical of the lower Borel subalgebra, and

$$\epsilon = \sum_{j=1}^l e_{\alpha_j}. \tag{2.21}$$

In the case $\mathfrak{g} = \mathrm{sl}(n)$, the affine space $\epsilon + \mathfrak{g}_0^\perp$ coincides with the set of lower Hessenberg matrices, i.e. matrices of the form $L = b^{\mathrm{T}} + \epsilon$, where b is upper triangular and $\epsilon_{jk} = \delta_{j,k-1}$.

The corresponding equation (2.3) is called *the full Kostant Toda flow* and its restriction to the symplectic leaf containing elements of the form

$$L = \sum_{j=1}^{l} b_j h_j + \sum_{j=1}^{l} e_{\alpha_j} + \sum_{j=1}^{l} a_j e_{-\alpha_j}, \tag{2.22}$$

where $a_j \neq 0$, $j = 1, \ldots, l$, is called the generalized Kostant Toda lattice.

Our key result here is that the Kostant Toda lattice equations are gradient on $\mathcal{J}_\Lambda$, an isospectral orbit of Jacobi elements through a generic element of the form (2.22).

Let $\mathfrak{n}^*$ be the nilradical of the upper Borel subalgebra and let $\pi_\mathfrak{n}$, $\pi_{\mathfrak{n}^*}$ be the projections of $\mathfrak{g}$ onto $\mathfrak{n}$ and $\mathfrak{n}^*$ in the decomposition $\mathfrak{g} = \mathfrak{n} + \mathfrak{h} + \mathfrak{n}^*$. Consider independent invariant polynomials $H_1, \ldots, H_l$ on $\mathfrak{g}$.

Lemma 2.5.

$$T_L \mathcal{J}_\Lambda = \operatorname{span}\{[L, \pi_\mathfrak{n} \nabla H_k(L)], k = 1, \ldots, l\}. \tag{2.23}$$

Proof. This follows from complete integrability. This is a set of commuting flows which span the tangent space to $\mathcal{J}_\Lambda$ at each point. □

We have:

Lemma 2.6. *For any two tangent vectors* $[L, \pi_\mathfrak{n} \nabla H_m(L)]$ *and* $[L, \pi_\mathfrak{n} \nabla H_n(L)]$, *the normal metric on* $\mathcal{J}_\Lambda$ *is given by*

$$\langle [L, \pi_\mathfrak{n} \nabla H_m(L)], [L, \pi_\mathfrak{n} \nabla H_n(L)] \rangle_N = \langle \pi_\mathfrak{n} \nabla H_m(L), \pi_{\mathfrak{n}^*} \nabla H_m(L) \rangle. \tag{2.24}$$

Proof. We have to show that the form on $T_L \mathcal{J}_\Lambda$ defined above is symmetric. Let σ be the antiautomorphism of $\mathfrak{g}$ such that $\sigma(e_{\alpha_j}) = e_{-\alpha_j}$, $\sigma(h_j) = h_j$. Since L has the form (2.22), it is possible to find $h \in \mathfrak{h}$ such that $\tilde{L} = \exp(\mathrm{ad}_h) L$ is stable with respect to σ. Note, however, that $\tilde{L}$ does not necessarily belong to $\mathfrak{g}_n$, the normal real form of $\mathfrak{g}$. Then $\sigma(\nabla H_m(\tilde{L})) = \nabla H_m(\tilde{L})$, $\pi_\mathfrak{n} \nabla H_m(\tilde{L}) = \pi_\mathfrak{n} \exp(\mathrm{ad}_h) \nabla H_m(L) = \exp(ad_h) \pi_\mathfrak{n} \nabla H_m(L)$ and $\pi_{\mathfrak{n}^*} \nabla H_m(\tilde{L}) = \exp(\mathrm{ad}_h) \pi_{\mathfrak{n}^*} \nabla H_m(L) = \sigma(\pi_\mathfrak{n} \nabla H_m(\tilde{L}))$. Therefore,

$$\begin{aligned}
&\langle \pi_\mathfrak{n} \nabla H_m(L), \pi_{\mathfrak{n}^*} \nabla H_n(L) \rangle \\
&\quad = \langle \pi_\mathfrak{n} \nabla H_m(\tilde{L}), \pi_{\mathfrak{n}^*} \nabla H_n(\tilde{L}) \rangle \\
&\quad = \langle \pi_\mathfrak{n} \nabla H_m(\tilde{L}), \sigma(\pi_\mathfrak{n} \nabla H_n(\tilde{L})) \rangle \\
&\quad = \langle \sigma(\pi_\mathfrak{n} \nabla H_m(\tilde{L})), \pi_\mathfrak{n} \nabla H_n(\tilde{L}) \rangle \\
&\quad = \langle \pi_\mathfrak{n} \nabla H_n(L), \pi_{\mathfrak{n}^*} \nabla H_m(L) \rangle. \qquad \square
\end{aligned}$$

It should be noticed that the form (2.24) is, generally speaking, indefinite. It is definite, however, if $\tilde{L} \in \mathfrak{g}_n$ or, in other words, if a_j in (2.3) are all positive.

We can now prove the following:

Theorem 2.7. *The Kostant Toda lattice flow X_H is a gradient flow on $\mathcal{J}_\Lambda$ with respect to the normal metric and the function $F = \langle L, N\rangle$.*

Proof. For an arbitrary tangent vector δL to $\mathcal{J}_\Lambda$, we need to show $\mathrm{d}F \cdot \delta L = \langle X_H, \delta L\rangle_N$. By Lemma 2.6, it is sufficient to consider $\delta L = [L, \pi_{\mathfrak{n}}\nabla H_m(\tilde{L})]$ for some m.

Now

$$\mathrm{d}H \cdot \delta L = \langle N, [L, \pi_{\mathfrak{n}}\nabla H_m(L)]\rangle = -\langle [L, N], \pi_{\mathfrak{n}}\nabla H_m(\tilde{L})\rangle.$$

On the other hand,

$$\langle X_H, \delta L\rangle_N = \langle [L, L_-], [L, \pi_{\mathfrak{n}}\nabla H_m(L)]\rangle_N = \langle L_+, \pi_{\mathfrak{n}}\nabla H_m(L)\rangle.$$

But $[L, N] = L_+ - L_-$ and hence

$$\langle [L, N], \pi_{\mathfrak{n}}\nabla H_m(L)\rangle = -\langle L_+, \pi_{\mathfrak{n}}\nabla H_m(L)\rangle.$$

This gives the result. □

We remark that this gives the Kostant Toda lattice flow as a gradient flow on a level set of its integrals with respect to the normal metric on a coadjoint orbit of the lower Borel algebra. In contrast, the result of Bloch et al. [5] described above shows that the symmetric Toda lattice flow is gradient on a level set of its integrals with respect to the normal metric on the compact Lie algebra in which it is naturally embedded.

2.3. *The full Kostant Toda flows*

To conclude this section, we establish the gradient nature of the higher full Kostant Toda flows (2.4) on $\mathfrak{g}$. Consider a manifold $\mathcal{O}^{\mathfrak{n}}_\Lambda = \{\mathrm{Ad}_g\Lambda : \mathrm{Ad}_g\Lambda \in \mathfrak{b}^* + \epsilon\}$. It is known [27] that $T_L\mathcal{O}^{\mathfrak{n}}_\Lambda = \{[L, \nu] : \nu \in \mathfrak{n}\}$. In particular, the higher Kostant Toda flows of the form

$$\dot{L} = [L, \pi_{\mathfrak{n}}\nabla H(L)], \tag{2.25}$$

where $H(L)$ is an invariant function on $\mathfrak{g}$, are tangent to $\mathcal{O}^{\mathfrak{n}}_\Lambda$.

We want to find a metric $B(\ ,\ )$ (possibly indefinite) on $\mathcal{O}^{\mathfrak{n}}_\Lambda$ such that the flows (2.25) are gradient with respect to $B(\ ,\ )$.

Let us fix a linear functional ϕ on $\mathfrak{h}$ and look for a metric defined on $T_L\mathcal{O}^{\mathfrak{n}}_\Lambda$ by

$$B([L, \nu_1], [L, \nu_2]) = \phi(\pi_{\mathfrak{h}}[B_L\nu_1, \nu_2]), \tag{2.26}$$

where B_L is a linear operator from $\mathfrak{n}$ to $\mathfrak{n}^*$. We need the following:

Lemma 2.8. *For invariant functions H_1, H_2,*

$$\phi(\pi_{\mathfrak{h}}[\pi_{\mathfrak{n}}\nabla H_1(L), \pi_{\mathfrak{n}^*}\nabla H_2(L)]) = \phi(\pi_{\mathfrak{h}}[\pi_{\mathfrak{n}}\nabla H_2(L), \pi_{\mathfrak{n}^*}\nabla H_1(L)]). \tag{2.27}$$

Proof. Due to the invariance of H_1, H_2, we have

$$\begin{aligned} 0 &= \pi_{\mathfrak{h}}[\nabla H_1(L), \nabla H_2(L)] \\ &= \pi_{\mathfrak{h}}([\pi_{\mathfrak{n}} \nabla H_1(L), \pi_{\mathfrak{n}^*} \nabla H_2(L)] \\ &\quad + [\pi_{\mathfrak{n}^*} \nabla H_1(L), \pi_{\mathfrak{n}} \nabla H_2(L)]). \qquad \square \end{aligned} \tag{2.28}$$

Lemma 2.8 shows that if we define B_L on the subspace $V \subset \mathfrak{n}$ spanned by $\{\pi_{\mathfrak{n}} \nabla H(L)\colon H$ is invariant$\}$ by

$$B_L \pi_{\mathfrak{n}} \nabla H(L) = \pi_{\mathfrak{n}^*} \nabla H(L), \tag{2.29}$$

then Eq. (2.26) defines a symmetric form on the subspace $[L, V] \subset T_L \mathcal{O}_\Lambda^{\mathfrak{n}}$.

Now consider an arbitrary extension of B_l from V to $\mathfrak{n}$, such that the form (2.26) is symmetric and let ∇_B be the gradient with respect to $B(\ ,\ )$. Note that if $B(\ ,\ )$ is positive definite on V, then the extension can be made positive definite too.

Lemma 2.9. *Eq.* (2.25) *describes a gradient flow* $\dot{L} = \nabla_B F_H$ *of the function*

$$F_H(L) = \phi(\pi_{\mathfrak{h}} \nabla H(L)). \tag{2.30}$$

Proof. For any $\delta L = [L, \nu] \in T_L \mathcal{O}_\Lambda^{\mathfrak{n}}$,

$$\begin{aligned} \delta F_H(L) &= \phi(\pi_{\mathfrak{h}}[\nabla H(L), \nu]) = \phi(\pi_{\mathfrak{h}}[\pi_{\mathfrak{n}^*} \nabla H(L), \nu]) \\ &= \phi(\pi_{\mathfrak{h}}[B_L \pi_{\mathfrak{n}} \nabla H(L), \nu]) = B([L, \pi_{\mathfrak{n}} \nabla H(L)], [L, \nu]). \qquad \square \end{aligned} \tag{2.31}$$

3. The geometry of the full Kostant and symmetric Toda lattices

3.1. Poisson maps

In this section we will be mainly concerned with the relationship between the full Kostant and full symmetric (signed) Toda flows in $\mathrm{sl}(n)$. Our main goal here is to construct a Poisson map from an open subset of $\epsilon + \mathfrak{b}_-$ to $S(J)$ that preserves the Toda flows, thus answering the question posed in [17]. Note that the obvious linear Poisson map coming from the identification $\epsilon + \mathfrak{b}_- \equiv S(J) \equiv \mathfrak{b}_+^*$ does not have the requisite property.

Let $L \in \epsilon + \mathfrak{b}_-$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $L = \mathrm{Ad}_C \Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and C can be chosen to be in the form $C = n\, v(\Lambda)$ with n lower triangular unipotent. Here $v(\Lambda)$ is the Vandermonde matrix corresponding to $\lambda_1, \ldots, \lambda_n$.

Assume now that $\lambda_1, \ldots, \lambda_n$ are real. Let us fix a signature matrix J and a diagonal matrix T (T is independent of L). We define an element $\beta = \beta(L)$ of the group of upper triangular matrices by

$$\beta^* J \beta = C^{-1*} T J C^{-1}, \tag{3.1}$$

provided the factorization (3.1) is possible. Denote

$$\mathcal{L} = \phi(L) = \mathrm{Ad}_\beta L. \tag{3.2}$$

Theorem 3.1. *The image under ϕ of the higher Toda flow* (2.4)

$$\dot{L} = [(L^k)_-, L] \tag{3.3}$$

on lower Hessenberg matrices is the higher Toda flow $\dot{\mathcal{L}} = [\pi_J(\mathcal{L}^k), \mathcal{L}]$ on $S(J) = \{X \in \mathrm{sl}(n)\colon JX = X^*J\}$, where π_J is the projection onto $\mathrm{so}(J)$ along the Borel subalgebra $\mathrm{so}(J)$ of J-orthogonal matrices.

Proof. From (3.1),

$$J\mathcal{L} = J\mathrm{Ad}_{\beta C}\Lambda = \mathrm{Ad}^{-1}{}_{C^*\beta^*}\mathrm{Ad}_{JT}\Lambda J = \mathrm{Ad}^{-1}{}_{\beta^*}L^* = \mathcal{L}^*J, \tag{3.4}$$

therefore $\mathcal{L} \in S(J)$.

Now assume that L evolves according to the equation $\dot{L} = [\nu, L]$, where ν is strictly lower triangular or, equivalently, $\dot{C} = \nu C$. Then

$$\dot{\mathcal{L}} = [\mathrm{Ad}_\beta \nu + \dot{\beta}\beta^{-1}, \mathcal{L}]. \tag{3.5}$$

Eq. (3.1) implies that $J(\mathrm{Ad}_\beta\nu + \dot{\beta}\beta^{-1}) + (\mathrm{Ad}_\beta\nu + \dot{\beta}\beta^{-1})^*J = \beta^{-1^*}(C^{-1^*}TJC^{-1}\nu + \nu^*C^{-1^*}TJC^{-1} + (C^{-1^*}TJC^{-1})^{\cdot})\beta^{-1} = 0$, i.e. $\mathrm{Ad}_\beta\nu + \dot{\beta}\beta^{-1} \in \mathrm{so}(J)$.

Since $\mathcal{L}^k = \mathrm{Ad}_\beta L^k$, we have for $\nu = (L^k)_-$,

$$\mathrm{Ad}_\beta\nu + \dot{\beta}\beta^{-1} = \mathcal{L}^k - (\mathrm{Ad}_\beta(L^k)_+ - \dot{\beta}\beta^{-1}) = \pi_J(\mathcal{L}^k). \qquad \square \tag{3.6}$$

If $J = \mathrm{Id}$ and T is an arbitrary diagonal matrix with positive coefficients, then β in (3.1) is always well-defined. Moreover, in this case the image under ϕ of the flow (3.3) will be the symmetric higher Toda flow which is known to be complete. Thus, one can view the map defined by (3.1) and (3.3) as a regularization map: the flow (3.3) with initial data L_0 which a priori can have finite time blowups is mapped into the complete symmetric flow with initial data $\mathcal{L}_0 = \phi(L_0)$. Blowups in (3.3) then correspond to the moments of time when $\mathcal{L}(t)$ leaves the image of ϕ. Note also that since β in (3.1) and (3.2) is upper triangular, ϕ maps tridiagonal flows into tridiagonal.

We now want to modify the definition of β by letting the diagonal matrix T depend on L and then see how the resulting map ϕ behaves with respect to Poisson structures on $\epsilon + \mathfrak{b}_-$ and $S(J)$ defined by (2.2).

First, notice that if T depends on L, then a calculation similar to the one in the proof of Theorem 3.1 shows that $\mathrm{Ad}_\beta\nu + \dot{\beta}\beta^{-1} - \frac{1}{2}f_T(\mathcal{L}) \in \mathrm{so}(J)$, where $f_T(\mathcal{L}) = \mathrm{Ad}_{\beta C}(T^{-1}\dot{T})$ commutes with $\mathcal{L}$. Then

$$\dot{\mathcal{L}} = [\pi_J(\mathcal{L}^k - \tfrac{1}{2}f_T(\mathcal{L})), \mathcal{L}]. \tag{3.7}$$

Next, we prove the following:

Lemma 3.2. *Let f_1, f_2 be restrictions to $S(J)$ of two Ad_{B_+}-invariant functions. Then*

$$\{f_1, f_2\}_{S(J)} \circ \phi = \{f_1 \circ \phi,\ f_2 \circ \phi\}_{\mathfrak{b}_- + \epsilon}. \tag{3.8}$$

Proof. Since $f_i, i = 1, 2$ are Ad_{B_+}-invariant, we have

$$\delta(f_i \circ \phi) = \langle \nabla f_i(\mathcal{L}), \mathrm{Ad}_\beta \delta L - [\mathcal{L}, \delta\beta\beta^{-1}] \rangle = \langle \nabla f_i(\mathcal{L}), \mathrm{Ad}_\beta \delta L \rangle.$$

Therefore, $\nabla f_i \circ \phi(L) = \mathrm{Ad}_\beta^{-1} \nabla f_i(\mathcal{L})$.

Moreover, analogous to (2.5), we have

$$\{f_1, f_2\}_{S(J)}(\mathcal{L}) = \langle \mathcal{L}, [\nabla f_1(\mathcal{L}), \nabla f_2(\mathcal{L})] \rangle.$$

Thus,

$$\begin{aligned}\{f_1 \circ \phi\ ,\ f_2 \circ \phi\}_{\mathfrak{b}_- + \epsilon} &= \langle L, [\mathrm{Ad}_\beta^{-1} \nabla f_1(\mathcal{L}), \mathrm{Ad}_\beta^{-1} \nabla f_2(\mathcal{L})] \rangle \\ &= \{f_1, f_2\}_{S(J)}(\phi(L)). \qquad \square\end{aligned}$$

Let us consider a particular choice of L-dependent T in (3.1):

$$T = \mathrm{diag}(\rho_1^{-1}, \ldots, \rho_n^{-1})\ J, \tag{3.9}$$

where the functions $\rho_1 = \rho_1(L), \ldots, \rho_n = \rho_n(L)$ are defined as coefficients in the decomposition

$$((\lambda - L)^{-1})_{11} = \sum_{j=1}^{n} \frac{\rho_j}{\lambda - \lambda_j}. \tag{3.10}$$

In the tridiagonal case $\rho_1, \ldots, \rho_n$ are the so-called Moser coordinates [29]. Their importance is based on the fact that the data $\{\lambda_1, \ldots, \lambda_n; \rho_1, \ldots, \rho_n\}$ determine a tridiagonal matrix L uniquely, and moreover,

$$\lambda_i,\ \mu_i = \log\left(\frac{\rho_i \prod_{j \neq i}(\lambda_i - \lambda_j)}{\rho_n \prod_{j \neq n}(\lambda_n - \lambda_j)}\right), \quad i = 1, \ldots, n-1, \tag{3.11}$$

provide action–angle variables for the Toda lattice.

In the general case, $\lambda_i, i = 1, \ldots, n-1$, do not form a maximal family of involutive integrals, but it is still true (cf. [13]) that (3.11) is a set of canonically conjugate functions both on $S(J)$ and $\epsilon + \mathfrak{b}_-$. Note also that, considered as functions on $\mathrm{sl}(n)$, ρ_i are invariant under the adjoint action of $\mathrm{SL}(n-1)$ imbedded into $\mathrm{SL}(n)$ as a right lower block.

If L evolves according to $\dot{L} = [(L^k)_-, L]$ then it is known that

$$\dot{\rho}_j(L) = -(\lambda_j^k - (L^k)_{11})\rho_j(L).$$

It follows that $\pi_J(f_T(\mathcal{L})) = \pi_J(\mathrm{Ad}_{\beta C}(T^{-1}\dot{T})) = \pi_J(\mathcal{L}^k)$, and therefore,

$$\dot{\mathcal{L}} = [\tfrac{1}{2}\pi_J(\mathcal{L}^k)), \mathcal{L}].$$

Lemma 3.3. *If T is given by* (3.9), *then*

$$\rho_j(L) = \rho_j(\mathcal{L}), \quad j = 1, \ldots, n. \tag{3.12}$$

Proof. Since $C = nv(\Lambda)$ and $L = \mathrm{Ad}_C \Lambda$, (3.6) implies

$$\rho_j(L) = C_{1j}(C^{-1})_{j1} = (C^{-1})_{j1}. \tag{3.13}$$

But this means that the first column of TJC^{-1} has all entries equal to 1, and therefore, coincides with the transpose of the first row of C. Then, by (3.1), all off-diagonal entries of the first column of $\beta^* J\beta$ are zero and thus the only nonzero entry of the first row of β is $(1, 1)$ entry and the statement follows. □

We call a matrix M generic if all its $k \times k$ left lower minors ($k = 1, \ldots, [n/2]$) are nonzero (cf. [13,17]). We have the following theorem (we remark that in the tridiagonal case this result reduces to the usual conjugation by diagonal matrices of the Kostant flow to the signed symmetric matrices, which is known to be symplectic):

Theorem 3.4. *Let T be defined by* (3.9) *and assume that for generic $L_0 \in \epsilon + \mathfrak{b}_-$, the factorization* (3.1) *is well-defined. Then in the neighborhood of L_0, the map $\phi : \epsilon + \mathfrak{b}_- \to S(J)$ is Poisson.*

Proof. To prove the statement, it is sufficient to check the identity (3.8) for a fixed given system of coordinate functions on $S(J)$. The system we are going to choose can be defined in the same way both for $S(J)$ and $\epsilon + \mathfrak{b}_-$. It will consist of functions (3.11) and restrictions of Ad_{B_+}-invariant functions. This will enable us to use Lemmas 3.2 and 3.3 to finish the proof. In fact Lemma 3.3 shows that the partial sets of action–angle variables (3.11) are the same for L and $\mathcal{L}$. The same is clearly true for the Ad_{B^+}-invariant functions. Below we show that these two sets of coordinates provide a complete set of coordinates.

For any generic M, there exists an upper-triangular matrix $\Gamma = \Gamma(M)$ such that

$$\mathrm{Ad}_\Gamma M = \sum_{k=0}^{[n/2]-1} e_{n-k,k} + \sum_{j=0}^{[n/2]-1} x_j e_{n-j,n-j} + U, \tag{3.14}$$

where U is a strictly upper triangular matrix (see, e.g. [1]).

Recall that symplectic leaves for (2.2) are orbits of the coadjoint action of B_+ and that the dimension of the orbit through a generic element is equal to $n(n+1)/2 - [(n+1)/2]$. If $M = L \in \epsilon + \mathfrak{b}_-$, the functions $x_j = x_j(L)$ are coadjoint invariants of B_+. The stabilizer G_0 of $\mathrm{Ad}_\Gamma L$ under this action of B_+ consists of diagonal matrices D, satisfying $d_{jj} = d_{n-j,n-j}$.

Let $U = (u_{ij}, i < j)$. Then the monomials

$$y_i = u_{i,n-i+1}, \quad i = 1, \ldots, [n/2], \tag{3.15}$$

$$y_{ij} = u_{ij} u_{j,n-i+1}, \quad i < j < n - i + 1, \tag{3.16}$$

$$y'_{ij} = u_{ij} u_{n-j+1,n-i+1}, \quad i < j, \tag{3.17}$$

$$y_{ijk} = u_{ij} u_{ki} u_{j,n-k+1}, \quad i < j, k < \min(i, n - j + 1), \tag{3.18}$$

are invariant under the adjoint action of the stabilizer G_0. This means that we can view $y_i, y_{ij}, y'_{ij}, y_{ijk}$ as restrictions to $\epsilon + \mathfrak{b}_-$ of Ad_{B_+}-invariant functions on $\mathrm{sl}(n)$.

In particular, we can choose among these functions $n(n-1)/2-[(n+1)/2]+1$ independent ones, e.g. y_{ij1}, $1<i<j<n$; y_{1j}, $j=2,\dots,n-1$; y'_{1j}, $j=2,\dots,[n/2]$.

For an orbit through L_0, which is determined by the fixed values of x_j, $j=0,\dots,[n/2]$, we choose now the system of coordinate functions

$$\begin{aligned}&\mu_j,\lambda_j,\quad j=1,\dots,n-1,\\ &z_k=z_k(U),\quad k=2n+1,\dots,n(n+1)/2-[(n+1)/2],\end{aligned}\tag{3.19}$$

where z_k are independent functions of $y_i, y_{ij}, y'_{ij}, y_{ijk}$, such that they do not depend on u_{1j}, $j=2,\dots,n$. Then off-diagonal elements of the first column $\nabla z_k(L)$ are all zero, and since μ_j are $SL(n-1)$-invariant functions, we have

$$\begin{aligned}\{z_k,\mu_j\}_{\epsilon+\mathfrak{b}_-}(L)&=\langle L,[\pi_+\nabla z_k(L),\pi_+\nabla\mu_j(L)]\rangle\\ &=\langle L,[\pi_+\nabla\mu_j(L),\pi_-\nabla z_k(L)]\rangle\\ &=\langle L,[\nabla\mu_j(L),\pi_-\nabla z_k(L)]\rangle=0.\end{aligned}\tag{3.20}$$

Note now that coordinates (3.19) can be defined on $S(J)$ in absolutely the same way. Furthermore, due to Lemma 3.3 and Ad_{B_+}-invariance of λ_j and z_k, we have

$$\mu_j\circ\phi=\mu_j,\quad \lambda_j\circ\phi=\lambda_j,\quad z_k\circ\phi=z_k.$$

Then the statement of the theorem follows from Lemma 3.2 and the fact that λ_j,μ_j, $j=1,\dots,n-1$, are canonically conjugate. □

3.2. Noncommutative integrability

The next question we would like to address is the interplay between the Hamiltonian and gradient behavior of Toda flows. As explained above and in [5], in the tridiagonal symmetric case, we have a gradient flow on the level set of the integrals, which is diffeomorphic to $\mathbb{R}^{n-1}$. However, as was shown in [13], level sets of the maximal Poisson commuting family containing higher Hamiltonians for full symmetric Toda flows in $\mathrm{sl}(n)$ are, in general, cylinders. In principle, this allows quasiperiodic behavior of the higher Toda flows. This possibility, though, is ruled out by the well-known asymptotic properties of the flows, which do not change with a transition from the tridiagonal to full case and which are the main reasons one might look for a gradient structure in the full symmetric or Kostant Toda flows.

The key to the explanation of this phenomenon from the Hamiltonian point of view is that there are many distinct maximal Poisson commutative families for the full Toda flows (this was first observed in [17] for the sl(4) case) and that one has to consider the level set of *all* the integrals. This level set is preserved under all higher Toda flows and diffeomorphic to $\mathbb{R}^{n-1}$ due to the *noncommutative integrability* of the Toda flow on a generic coadjoint orbit.

Indeed, in the proof of Theorem 3.4 we constructed the family (3.15)–(3.18) of Ad_{B_+}-invariant functions whose restriction to the generic coadjoint orbit in $\epsilon+\mathfrak{b}_-$ provides a

(Poisson-noncommutative) family of independent integrals for the Toda flow. The number of integrals in this family is equal to the dimension of the orbit minus the rank of $\mathrm{sl}(n)$. Furthermore, the invariant polynomials of L can, from (3.14), be expressed via the integrals (3.15)–(3.18) and are in involution with any of them. This puts us into the framework of Nehoroshev's theorem:

Theorem 3.5 (Nehoroshev [30]). *If a Hamiltonian system on a $2n$-dimensional symplectic manifold possesses $2n-k$ independent first integrals $F_1, \dots, F_{2n-k}$ such that $F_1, \dots, F_k$ are in involution with all F_i, $i = 1, \dots, 2n-k$, then all trajectories lie on k-dimensional invariant tori or cylinders.*

The same phenomenon can be observed for the Toda flow on generic coadjoint orbits in an arbitrary semisimple Lie algebra $\mathfrak{g}$. Indeed, it follows from unpublished results by Kostant on the structure of generic coadjoint orbits, that any generic element in $\epsilon + \mathfrak{b}_-$ can be brought to a normal form analogous to (3.14). Namely, let $\mathfrak{M} = \{\beta_1, \dots, \beta_r\}$ be the maximal set of of *strongly orthogonal* positive roots, i.e. maximal subset of positive roots, such that for any two roots in it neither their sum nor their difference is a root. A description of $\mathfrak{M}$ via Kostant's "cascade construction" can be found, e.g. in [28].

Let O_L be a generic coadjoint orbit O_ζ through $L \in \epsilon + \mathfrak{b}_-$. Then the following result is true.

Theorem 3.6 (Kostant, unpublished note – see [22]). *Let $\mathfrak{h}_0$ be an orthogonal complement in $\mathfrak{h}$ to* $\mathrm{Span}\{[e_{-\beta}, e_\beta], \beta \in \mathfrak{M}\}$. *If $L \in \epsilon + \mathfrak{b}_-$ is generic, then there exists a unique element $h_0 \in \mathfrak{h}_0$ such that*

$$L_0 = \sum_{i=1}^{r} e_{-\beta_i} + h_0 + \epsilon \in O_L. \tag{3.21}$$

The codimension of O_L in $\epsilon + \mathfrak{b}_-$ is equal to the dimension of the stabilizer of L_0 and is $l - r$.

Theorem 3.6 allows us to establish noncommutative integrability of the Toda flows on generic coadjoint orbits, which, in turn, makes Nehoroshev's theorem applicable in this case.

Theorem 3.7. *The Poisson subalgebra of first integrals of the Toda flow on O_L is generated by restrictions to O_L of Ad_{B_+}-invariant functions on $\mathfrak{g}$ and has functional dimension* $\dim O_L - l$. *Its center is generated by restrictions of the Chevalley invariants of $\mathfrak{g}$.*

We give a sketch of the proof of Theorem 3.7. Details can be found in the forthcoming paper by Gekhtman and Shapiro [22].

Proof. By Theorem 3.6, for any generic $L \in \epsilon + \mathfrak{b}_-$ there exists $b_L \in B_+$ such that $\mathrm{Ad}^*_{b_L} L = L_0 = \sum_{i=1}^{r} e_{-\beta_i} + h_0 + \epsilon$, where $h_0 \in \mathfrak{h}_0$. Denote $\mathfrak{h}_1 = \mathfrak{h}_0^\perp$ and consider the factorization $\mathbf{T} = \mathbf{T}_0\mathbf{T}_1$ of the maximal torus $\mathbf{T}$ corresponding to the linear space decomposition $\mathfrak{h} = \mathfrak{h}_0 + \mathfrak{h}_1$. Then the stabilizer of L_0 under the coadjoint action is $\mathbf{T}_0$ and b_L is defined uniquely up to a right multiplication by elements of $\mathbf{T}_0$. Let $b_L = \tilde{b}_L t_L$ be a factorization of b_L into the product of unipotent $\tilde{b}_L$ and $t_L \in \mathbf{T}$. We can make the choice of b_L unique by demanding that t_L belongs to $\mathbf{T}_1$.

For a generic element $g \in \mathfrak{g}$ there exists a unique element $L \in \epsilon + \mathfrak{b}_-$ such that $\pi_{b_-}(g - L) = 0$. Define now b_g to be equal to b_L as defined above. Note that if β is an element from B_+ and $\beta = \tilde{\beta}\, t_0 t_1$ is its factorization into the product of unipotent $\tilde{\beta}$, $t_0 \in \mathbf{T}_0$ and $t_1 \in \mathbf{T}_1$, then

$$b_{\mathrm{Ad}_\beta g} = (\tilde{\beta} t_0 t_1 \tilde{b}_L t_0^{-1} t_1^{-1})\, t_1 t_L = \beta b_g t_0^{-1}. \tag{3.22}$$

For any positive root α, define a function

$$\varphi_\alpha(g) = \langle g, \mathrm{Ad}_{b_g} e_{-\alpha}\rangle. \tag{3.23}$$

It can be shown that functions φ_α are independent of O_L.

It follows from (3.22) that $\varphi_\alpha(g)$ is semiinvariant under the adjoint action of B_+:

$$\varphi_\alpha(\mathrm{Ad}_\beta g) = \langle g, \mathrm{Ad}_{b_g} \mathrm{Ad}_{t_0}^{-1} e_{-\alpha}\rangle = \chi_\alpha(\beta)\, \varphi_\alpha(g).$$

Moreover, if $\mathbf{k}$ is an integral vector such that $\nu = \sum_{\alpha\in\Phi^+} k_\alpha \alpha,\ k_\alpha \in \mathbb{Z}$, annihilates $\mathfrak{h}_0$ then a function

$$\theta^{\mathbf{k}}(g) = \Pi_{\alpha\in\Phi^+}(\varphi_\alpha(g))^{k_\alpha} \tag{3.24}$$

is Ad_{B_+}-invariant on $\mathfrak{g}$.

The number of linearly independent vectors $\mathbf{k}$ such that ν annihilates $\mathfrak{h}_0$ is equal to the number of positive roots minus the rank of the matrix $(\alpha(\eta_i))_{\alpha\in\Phi^+}$, where $\eta_i,\ i = 1, \ldots, \dim \mathfrak{h}_0$, is a basis of $\mathfrak{h}_0$. Clearly, this rank is equal to $\dim \mathfrak{h}_0 = l - r$. Thus the number of independent functions on O_L, $\theta^{\mathbf{k}}$, is equal to $\dim O_L - l$. On the other hand if H is an invariant polynomial on $\mathfrak{g}$, then $H(L) = H(\mathrm{Ad}_{b_L}^{-1} L)$ can be expressed in terms of the functions $\theta^{\mathbf{k}}$ and is in involution with any of them. □

To conclude this section, we compare the behavior of the full Kostant Toda flows with that of more general Toda flows defined on the whole space of $n \times n$ matrices, namely, the QR-flows that were extensively studied and shown to be completely integrable by Deift et al. [14]. The evolution in this case is given by the Lax equation

$$\dot{M} = [M, M_- - M_-^{\mathrm{T}}], \tag{3.25}$$

where M_- is a strictly lower triangular part of M. Note that we have applied a transposition to the equation originally considered in [14].

It is known that if the initial data $M(0)$ belong to the open set of elements with distinct real eigenvalues in the space of real $n \times n$ matrices, then as t tends to ∞, $M(t)$ tends to an upper triangular matrix with diagonal entries arranged in ascending order. This (a) indicates a gradient-like behavior and (b) suggests that as in the case of full Kostant Toda flows, invariant manifolds have dimension $(n-1)$ which is much smaller than half the dimension of a generic symplectic leaf, equal in this case to $n(n-1)$. Recall that the maximal Poisson commuting family constructed in [14] contains two subfamilies of integrals: one coincides with the maximal family of B-invariant Poisson commuting integrals for the generic symmetric or Kostant Toda lattice and the other consists of functions invariant under the conjugation by elements of the orthogonal group. Elements of the latter subfamily generate periodic flows.

Our first remark here is that, as in Eqs. (3.1) and (3.2), the flows of (3.25) can be conjugated to, generally speaking, linear combinations of higher symmetric Toda flows. We shall not discuss here how to make a conjugation map Poisson, but limit ourselves to establishing noncommutative integrability of (3.25).

Theorem 3.8. *The Poisson subalgebra of first integrals of the Toda flow* (3.25) *on generic symplectic leaves has functional dimension* $(n-1)^2$. *Its center is generated by the Chevalley invariants of* $\mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^n)$.

Proof. We refer to Deift et al. [14] for the fact that if a function f_1 (resp. f_2) is invariant under the conjugation by elements of the upper triangular (resp. orthogonal) group, then f_1 and f_2 are in involution and both f_1 and f_2 are in involution with any Chevalley invariant.

Let $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$) denote the subalgebra of functions on $gl(n)$ invariant under the conjugation by elements of the upper triangular (resp. orthogonal) group. We refer to Deift et al. [14] for the fact that if $f_1 \in \mathfrak{A}_1$ and $f_2 \in \mathfrak{A}_2$, then f_1 and f_2 are in involution and both f_1 and f_2 are in involution with any Chevalley invariant. Note also that the family of independent Casimir functions that defines generic symplectic leaves belongs to $\mathfrak{A}_1 + \mathfrak{A}_2$.

Thus, it is sufficient to show that the functional dimension of the subalgebra of functions on $\mathrm{gl}(n)$ generated by $\mathfrak{A}_1, \mathfrak{A}_2$ is $n^2 - n$. By Theorem 3.7, $\mathfrak{A}_1$ is generated by $n(n-1)/2$ integrals and Casimirs. On the other hand, $\mathfrak{A}_2$ is generated by matrix elements of the upper triangular factor U in the Schur decomposition $M = OUO^{\mathrm{T}}$, so its functional dimension is $n(n+1)/2$. Since $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is generated by the Casimir function $\mathrm{Tr}(M)$ and the Chevalley invariants $\mathrm{Tr}(M^2), \ldots, \mathrm{Tr}(M^n)$, the statement follows. □

4. Conclusions

In this paper we have proved a number of properties describing the qualitative behavior of a fairly general class of Toda flows. We conclude by mentioning some related issues.

While the symmetric flows are known to have long time existence, it is known that this is not true in general in the nonsymmetric case (either tridiagonal or full). Kodama and Ye [24], Brockett and Faybusovich [10], and Faybusovich [18] for example showed that the signed Toda flows may experience a blow up in a finite time. On the other hand, Gekhtman and Shapiro [21] found necessary and sufficient conditions for completeness of the Kostant Toda flows. Similar conditions may be found for other flows discussed here. For example the result of Gekhtman and Shapiro may be extended in order to find a criterion for solution of signed Toda lattice to be everywhere nonsingular as follows:

Let the signature matrix S be equal to $\mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i = \pm 1$. Let L in (2.12) be tridiagonal and set $D = \mathrm{diag}(d_1, \ldots, d_n) = \mathrm{diag}(1, \epsilon_2 a_1, \epsilon_2\epsilon_3 a_1 a_2, \ \ldots, \ \epsilon_1 \cdots \epsilon_n a_0 \cdots a_n)$. Then $\tilde{L} = D^{-1}LSD$ has entries one on its subdiagonal and the $(i, i+1)$ entry of $\tilde{L}$ is $\epsilon_i\epsilon_{i+1}a_i^2$.

Let $\lambda_1 > \cdots > \lambda_s$, $s \leq n$, be distinct eigenvalues of $L(0)S$ and $C(L(0))$ be the matrix whose columns constitute a Jordan basis of the upper triangular Jordan normal form of $L(0)S$. Then columns of $\tilde{C} = D^{-1}C(L(0))$ form a Jordan basis for $\tilde{L}(0)$.

Let $\mu_i = \mathrm{sign}(d_i(0))$. Denote by $C^{1,\ldots,k}_{i,\ldots,i+k-1}$, a minor of $C(L(0))$ formed from the k first columns and the ith, …, $(i+k-1)$th rows. Then one can show that flow (2.12) is complete if and only if

(1) all λ_i $(i = 0, \ldots, s)$ are real,

(2) for any $k = 1, \ldots, n-1$, all numbers $\mu_i \cdots \mu_{i+k-1}C^{1,\ldots,k}_{i,\ldots,i+k-1}$ $(i = 1, \ldots, n-k)$ are nonzero and have the same sign.

The geometry of convex polytopes has been very useful for understanding the qualitative behavior of the Toda flows. Van Moerbeke [37] and Deift et al. [15] observed that an isospectral manifold of Jacobi (symmetric, tridiagonal) matrices with nonnegative off-diagonal elements, i.e. a Toda orbit, is homeomorphic to a convex polytope. In [6], it is shown, by studying the Kähler geometry of the Toda flows, that this polytope is in fact the image of a momentum map. The usual tridiagonal flow thus remains in such a polytope.

Polytopes are also useful in considering the tridiagonal flows where the off-diagonal elements are not taken to be positive. Tomei [36] considered the manifold formed when one allows the off-diagonal elements of the Jacobi matrices to change sign, but nonetheless requires the matrix to remain symmetric. In other words, Tomei analyzed the topology of the set of real symmetric tridiagonal matrices with fixed eigenvalues. This turns out to be a smooth orientable manifold, but one that is closely related to a manifold formed by "glueing" polytopes of the type described above together. In fact this manifold (call it $M_{\lambda_1,\lambda_2,\ldots,\lambda_n}$) turns out to be a "small cover" of a permutohedron P_n in the sense of Davis [11] and Davis and Januszkiewicz [12]. Kodama and Ye [26] carried out a similar construction, but using a glueing rule derived from the indefinite Toda flows. The idea is to glue together isopectral sets of Jacobi matrices of fixed signature, and to use the dynamics of flows to give the glueing rule.

It would be interesting to pursue the role of convexity in understanding the Kostant and full Toda flows – we intend to pursue this idea in a future publication.

In conclusion, we note that we have given here a rather general description of the various kinds of finite Toda flow, symmetric and nonsymmetric, tridiagonal and full, indicated how to pass from one to another, and described their qualitative nature, including the relationship of their gradient to their Hamiltonian properties.

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